The Limit Laws

1. $\lim_{x \to a} x = a$
2. $\lim_{x \to a} c = c$

- Let $f(x)$ & $g(x)$ be defined for all $x \neq a$ over some open interval contain a.

If $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$

① $\lim_{x \to a} \left(f(x) \pm g(x) \right) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M.$

② $\lim_{x \to a} \left(c\, f(x) \right) = c \left(\lim_{x \to a} f(x) \right) = cL$

③ $\lim_{x \to a} \left[f(x) \cdot g(x) \right] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M.$

④ $\lim_{x \to a} \left(\dfrac{f(x)}{g(x)} \right) = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M}$ for $M \neq 0$.

⑤ $\lim_{x \to a} \left(f(x) \right)^n = \left(\lim_{x \to a} f(x) \right)^n = L^n$

⑥ $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ [Note: $\sqrt[m]{a} = a^{1/m}$]

Egs: ① $\lim_{x \to -3} (4x + 2) = \lim_{x \to -3} 4x + \lim_{x \to -3} 2$

$= 4 \cdot \lim_{x \to -3} x + 2 = 4 \cdot (-3) + 2 = -12 + 2$

$= -10$

② $$\lim_{x\to 2} \frac{2x^2-3x+1}{x^3+4} = \frac{\lim_{x\to 2}(2x^2-3x+1)}{\lim_{x\to 2}(x^3+4)}$$

$$= \frac{2\lim_{x\to 2} x^2 - 3\lim_{x\to 2} x + \lim_{x\to 2} 1}{\lim_{x\to 2} x^3 + \lim_{x\to 2} 4}$$

$$= \frac{2\cdot(\lim_{x\to 2} x)^2 - 3(\lim_{x\to 2} x) + (\lim_{x\to 2} 1)}{(\lim_{x\to 2} x)^3 + (\lim_{x\to 2} 4)}$$

$$= \frac{2\cdot(2)^2 - 3(2)+1}{2^3+4} = \frac{8-6+1}{8+4} = \frac{3}{12} = \frac{1}{4}$$

⊛ Note: $a^2 - b^2 = (a+b)(a-b)$, $a > b > 0$

⊛ Note: $(\sqrt{a}-\sqrt{b})\cdot(\sqrt{a}+\sqrt{b}) = (\sqrt{a})^2 + \cancel{\sqrt{a}\sqrt{b}} - \cancel{\sqrt{a}\sqrt{b}} - (\sqrt{b})^2$

$= a - b$, $a > b > 0$

Two ways to deal with $\frac{0}{0}$ situations

(1) Factoring & Cancelling [mostly works with algebraic functions]

③ Find limit at $x=2$ for $f(x) = \frac{x^2-2x}{x^2-4}$

$f(2) = \frac{0}{0}$ [indeterminate]

But apart from $x=2$, $f(x) = \frac{x(x-2)}{(x+2)(x-2)} = \frac{x}{x+2}$

Then $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x}{x+2} = \frac{2}{2+2} = \frac{1}{2}$

(2) Multiplying by Conjugates:

④ $$\lim_{x\to 5} \frac{\sqrt{x+4}-3}{x-5} \quad \left[\frac{0}{0},\ \text{Indeterminate}\right]$$

$$= \lim_{x\to 5} \frac{\sqrt{x+4}-3}{x-5} \cdot \frac{\sqrt{x+4}+3}{\sqrt{x+4}+3}$$

$$= \lim_{x\to 5} \frac{(x+4)-3^2}{(x-5)(\sqrt{x+4}+3)}$$

$$= \lim_{x\to 5} \frac{(x-5)}{(x-5)(\sqrt{x+4}+3)}$$

$$= \lim_{x\to 5} \frac{1}{\sqrt{x+4}+3} = \frac{1}{\sqrt{5+4}+3} = \frac{1}{6}$$

⑤ $$f(x) = \begin{cases} 4x-3, & \text{if } x<2 \\ (x-3)^2, & \text{if } x\geq 2 \end{cases}$$

Find: $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2} f(x)$

$$\lim_{x\to 2^-} f(x) = 5, \quad \lim_{x\to 2^+} f(x) = 1, \quad \lim_{x\to 2} f(x) = \text{DNE}$$

Squeeze Theorem: Let $f(x)$, $g(x)$ & $h(x)$ be defined for $x \neq a$.

If
$$\begin{array}{ccccc} f(x) & \leq & g(x) & \leq & h(x) \\ \Big\downarrow \lim_{x\to a} & & \Big\downarrow \lim_{x\to a} & & \Big\downarrow \lim_{x\to a} \\ L & & L & & L \end{array}$$

Eg. Find $\lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$.

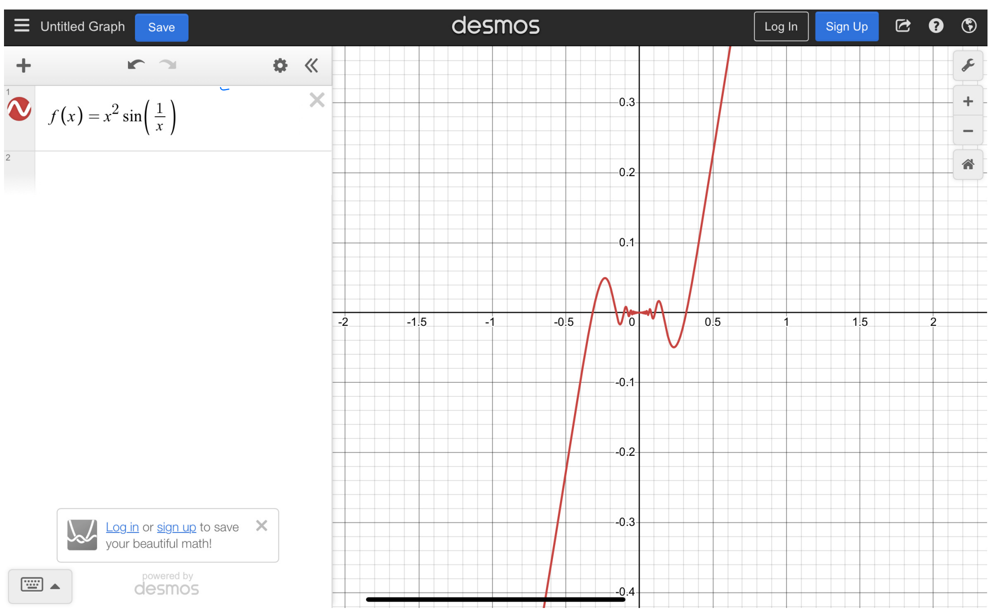
We know for $x \neq 0$, $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$, for all x.

$$\Rightarrow -x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2, \text{ for all } x$$

Now, $\lim_{x\to 0} -x^2 = -\left(\lim_{x\to 0} x\right)^2 = -0^2 = 0$

Similarly, $\lim_{x\to 0} x^2 = 0$

Then, $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.



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